# RREL REPORT

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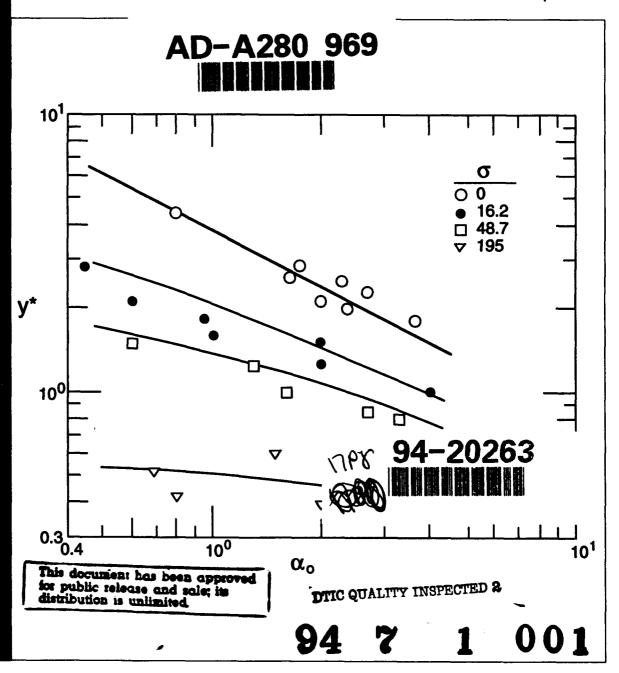




# Dependence of Segregation Potential on the Thermal and Hydraulic Conditions Predicted by Model M<sub>1</sub>

Yoshisuke Nakano

**April 1994** 



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The segregation potential (SP) model is semiempirical in nature. An accurate mathematical model is needed that provides the functional dependence of SP on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) der "bing the properties of a given soil. In response to such a need a mathematic model called  $M_1$  was introduced and efforts have been made to validate  $M_1$  with empirical findings and experimental data. In this report we will show that the functional dependence of SP on pertinent variables predicted by  $M_1$  is consistent with empirical findings that were used to build the SP model.

Cover: See Figure 5.

For conversion of SI metric units to U.S./British customary units of measurement consult Standard Practice for Use of the International System of Units (SI), ASTM Standard E380-89a, published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.

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### **PREFACE**

This report was prepared by Dr. Yoshisuke Nakano, Chemical Engineer, of the Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory. Funding was provided by DA Project 4A161102AT24, Research in Snow, Ice and Frozen Ground, Task SC, Work Unit F01, Physical Processes in Frozen Soil.

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### **NOMENCLATURE**

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$R_{10}$ region where $0 \ge T > T_1^{-1}$	-	· · · · · · · · · · · · · · · · · · ·		
	R <sub>10</sub>	region where $0 \ge T > T_1^{-1}$		

# Dependence of Segregation Potential on the Thermal and Hydraulic Conditions Predicted by Model M<sub>1</sub>

### YOSHISUKE NAKANO

### INTRODUCTION

We will consider the one-directional steady growth of an ice layer under an overburden pressure  $P_a$  ( $\geq$ 0). Let the freezing process advance from the top down and the coordinate x be positive upwards, with its origin fixed at some point in the unfrozen part of the soil. A freezing soil in this problem may be considered to consist of three parts: the unfrozen part  $R_0$ , the frozen fringe  $R_1$  and the ice layer  $R_2$ , as shown in Figure 1 where  $n_1$  is the interface between  $R_1$  and  $R_2$ ,  $n_0$  is the 0°C isotherm and n is the reference boundary. We will assume that the pressure of water is kept constant,  $P_n$  at n. The physical properties of parts  $R_0$ , and  $R_2$  are well understood but our knowledge of the

 $R_{2}$   $T = T_{1}$   $R_{1}$   $T = T_{0} = 0^{\circ}C$   $R_{0}$ 

Figure 1. Schematic drawing of a steadily growing ice layer in a freezing soil.

physical properties and the dynamic behavior of part  $R_1$  does not appear sufficient for engineering applications.

Konrad and Morgenstern (1980, 1981) empirically found that the mass flux of water f at the formation of the so-called final ice lens is proportional to the average temperature gradient  $T_f$  in  $R_1$ . This may be written as

$$SP = -f/T_t' \tag{1}$$

where a prime denotes differentiation with respect to x. The positive proportionality factor SP is termed the segregation potential, which is a property of a given soil.

Konrad and Morgenstern (1982b) also found empirically that SP depends on the applied pressure  $P_a$  and the pressure of water  $P_0$  at  $n_0$  and that SP is a decreasing function of  $P_a$  and the suction  $-P_0$ . This may be written as

$$SP = y \left( P_{0}, P_{a} \right) \tag{2}$$

$$\frac{\partial}{\partial P_0} y > 0 , \frac{\partial}{\partial P_0} y < 0.$$
 (3)

Extending the concept of segregation potential, Konrad and Morgenstern (1982a) introduced a semiempirical model (SP model) of soil freezing.

Several researchers have used the SP model to analyze frost heave data from field and laboratory tests (Nixon 1982, Knutsson et al. 1985, Jessberger and Jagow 1989). Their results suggest that the accuracy of the SP model suffices engineering needs. However, the SP model is not immune to criticism. Some limitations or shortcomings of the SP model have appeared (Ishizaki and Nishio 1985, Van Gassen and Sego 1989, Nixon 1991).

Nevertheless, it is important to point out that an accurate model of soil freezing, if it exists, should be able to explain important empirical findings such as eq 1, 2 and 3.

Recently the steady growth condition of an ice layer was studied mathematically and experimentally under a negligible applied pressure (Nakano 1990, Takeda and Nakano 1990, Nakano and Takeda 1991) and under an applied pressure (Takeda and Nakano 1993, Nakano and Takeda 1994). They have shown that a model called M<sub>1</sub> accurately describes the experimentally determined condition of steady growth.

For the problem of a steadily growing ice layer, the  $M_1$  model is defined as a frozen fringe where ice may exist but does not grow, and where the mass flux of water f is given as

$$f = -K_1 \frac{\partial P}{\partial x} - K_2 \frac{\partial T}{\partial x}$$
 for  $x$  in  $R_1$  (4a)

$$K_2/K_1 \to \gamma$$
 as  $f \to 0$  (4b)

$$\lim_{x \to n_1} P(x) = P_a$$

$$x \to n_1$$

$$x \text{ in } R_1$$
(4c)

where  $\gamma$  is a constant, P is the pressure of unfrozen water and  $K_i$  (i=1,2) is the transport property of a given soil that generally depends on the temperature and the composition of the soil. The  $M_1$  model is a generalization of somewhat simpler models (Derjaguin and Churaev 1978, Ratkje et al. 1982, Kuroda 1985, Horiguchi 1987) in which the ratio  $K_2/K_1$  is equal to  $\gamma$  regardless of f.

Experimental methods were proposed to determine  $K_1$  (Williams and Burt 1974, Horiguchi and Miller 1983) and  $K_2$  (Perfect and Williams 1980). According to Horiguchi and Miller (1983)  $K_1$  of several frozen porous media is described in the general form given as

$$K_1 = A_0 |T|^B \tag{5}$$

where  $A_0$  and B are positive material constants. Nakano and Takeda (1994) experimentally determined  $K_2$  of Kanto loam in the form given as

$$K_{2} = \begin{cases} K_{20} & A \leq T < 0 \\ K_{20}|A/T|^{B} & A > T \end{cases}$$
 (6)

where  $K_{20}$  and B are positive constants and A is a small negative constant. Since empirically determined functions  $K_1$  and  $K_2$  are known to be bound-

ed, the functional form of eq 6 is a better approximation to their actual behaviors in a neighborhood of  $T = 0^{\circ}$ C.

The SP model is semiempirical in nature. An accurate mathematical model is needed that provides the functional dependence of SP on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) describing the properties of a given soil. In this report we will present the results of our study on the problem of a steadily growing ice layer by using the  $M_1$  model. We will show that the  $M_1$  model provides the functional dependence of SP that is consistent with empirical findings (eq 1, 2 and 3).

### PROPERTIES OF M<sub>1</sub>

An analytical solution was derived to the problem of a steadily growing ice layer under the  $M_1$ model (Nakano 1990). A schematic drawing of a typical temperature profile is shown in Figure 2, where the temperature is continuous but the temperature gradient is discontinuous at  $n_1$  where the ice layer grows. Since the amount of heat transported by convection is generally much less than that by conduction, the temperature profiles in  $R_0$ , and  $R_2$  in the vicinity of  $R_1$  are nearly linear (Nakano and Takeda 1991).

The temperature  $T_1$  and the limiting value of the temperature gradient at  $n_1$  are given approximately (Nakano 1990) as

$$T_1 = -a \alpha_0 \delta \tag{7a}$$

$$T'(n_1^+) = -b \alpha_0 \tag{7b}$$

where  $\alpha_0$  is the absolute value of the temperature gradient at  $n_0$ ,  $\delta$  is the thickness of  $R_1$  and  $n_1$ <sup>+</sup> denotes the limiting value as x approaches  $n_1$  while x is in  $R_1$ . The a and b in eq 7a and 7b are correction factors that take account of effects of the convective heat transport and the variable thermal conductivity in  $R_1$  (Nakano and Takeda 1991). When these effects are negligible, a and b are equal to one. We will neglect these effects hereafter.

When an ice layer is steadily growing, according to the  $M_1$  model, ice does not grow in  $R_1$ ; the growth of ice occurs only at  $n_1$ . The balance of heat for  $R_1$  is given as

$$k_1\alpha_1 = k_0\alpha_0 + (L - c_2 T_1)f$$
 (8)

where  $k_1$  and  $k_0$  are the thermal conductivities of  $R_2$ 

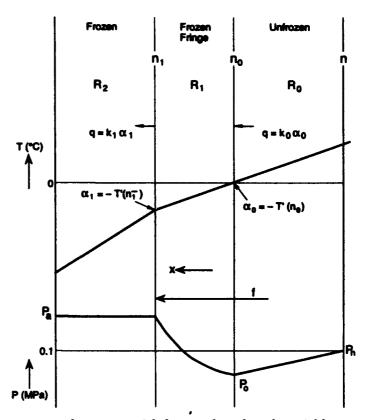


Figure 2. Schematic drawing of typical temperature and pressure profiles according to the  $M_1$  model.

and  $R_0$ , L is the latent heat of fusion of water, and  $c_2$  is the heat capacity of  $R_2$ .  $\alpha_1$  is the absolute value of the limiting temperature gradient at  $n_1$  defined as

$$\alpha_1 = -\lim_{\substack{x \to n_1 \\ x \text{ in } R_2}} T'(x) = -T'\left(n_1^{-}\right) \tag{9}$$

where  $n_1^-$  denotes the limiting value as x approaches  $n_1$  while x is in  $R_2$ .

A schematic drawing of a typical pressure profile is given also in Figure 2 where  $P_n$  is kept at 0.1 MPa. The pressure of water P is continuous in the combined region of  $R_0$ , and  $R_1$  but the gradient of P may be discontinuous at  $n_0$ . Darcy's law clearly holds true in  $R_0$ . Neglecting the gravity effect, we obtain

$$\Delta P = P_{\rm n} - P_0 = \delta_0 K_0^{-1} f \tag{10}$$

where  $\delta_0 = n_0 - n > 0$ .

It has been shown (Nakano 1992) that according to the  $M_1$  model, the segregation potential function y is given as

$$y = -f^*/T^*(n_1^+) = f^*\alpha_0^{-1} = K_2(T_1^*)$$
 (11a)

where an asterisk denotes the value of a variable at the formation of the final ice lens. For given  $\sigma$  and  $\delta_0$ , the necessary and sufficient condition for the steady growth of an ice layer is given (Nakano and Takeda 1991, Nakano and Takeda 1994) as

$$(k_1/k_0)\alpha_1 > \alpha_0 \ge k_1(k_0 + Ly)^{-1}\alpha_1.$$
 (11b)

The condition of eq 11b is explained in the diagram of temperature gradients (Fig. 3).  $R_i^{**}$  in Figure 3 is given as

$$\alpha_0 = (k_1/k_0)\alpha_1$$
 on  $R_1^{**}$ . (11c)

If an existing ice layer neither grows nor melts, f vanishes on  $R_i^{**}$ . We will refer to the region as  $R_m$  where  $\alpha_0 > (k_1/k_0)\alpha_1$ , f < 0 and an ice layer is melting.  $R_i^*$  is given as

$$\alpha_0 = k_1(k_0 + Ly)^{-1} \alpha_1$$
 on  $R_s^*$ . (11d)

The boundary  $R_s^*$  is a curve stemming from the origin because y depends on  $\alpha_0$ . The steady growth of an ice layer occurs in the region  $R_s$ . The formation of the final ice lens occurs on  $R_s^*$  where eq 11a holds true.

The temperature  $T_1$  is given (Nakano 1990, Nakano and Takeda 1994) as

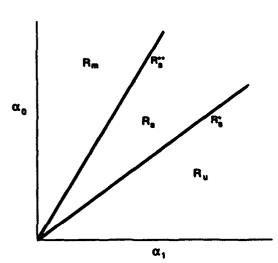


Figure 3. Diagram of temperature gradients  $\alpha_1$  and  $\alpha_0$ .

$$-T_1 = (\sigma + \delta_0 K_0^{-1} f) / I$$
 (12a)

$$I = \gamma \phi_{01} - K_0^{-1} f \alpha_0^{-1} \phi_{11}$$
 (12b)

where  $\sigma$ ,  $\phi_{01}$  and  $\phi_{11}$  are defined as

$$\sigma = P_{\mathbf{a}} - P_{\mathbf{n}} \tag{13a}$$

$$\phi_{01}(T_1) = T_1^{-1} \int_0^{T_1} (K_0/K_1) (K_2/K_{20}) dT$$
 (13b)

$$\phi_{11}(T_1) = T_1^{-1} \int_0^{T_1} (K_0/K_1) dT.$$
 (13c)

Using eq 11a, 12a and 12b, the mass flux  $f^*$  at the formation of the final ice lens is given as

$$f^* = (K_0/\delta_0) [-T_1^* I - \sigma]$$
 (14a)

$$I(T_1^*) = \gamma \phi_{01}(T_1^*) - K_0^{-1} K_2(T_1^*) \phi_{11}(T_1^*). \quad (14b)$$

Equations 14a and 14b imply that  $f^*$  is uniquely determined by  $T_1^*$  when  $\delta_0$  and  $\sigma$  are given. Using eq 11a, we will reduce eq 14a and 14b to:

$$K_2(T_1^*) = K_0(\alpha_0 \delta_0)^{-1} [-T_1^* I - \sigma].$$
 (15)

Since  $K_2$  is an increasing function of  $T_1^*$ ,  $K_2$  and  $T_1^*$  are one-to-one and  $T_1^*$  may be considered to be a function of  $K_2$ . Therefore, eq 15 implies that y is a function of  $\alpha_0$ ,  $\delta_0$  and  $\sigma$ .

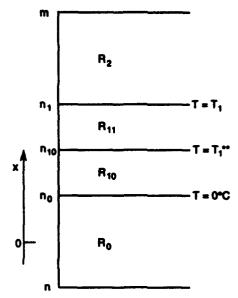


Figure 4. Schematic drawing of a steadily growing ice layer under an applied load.

$$y = y(\alpha_0, \delta_0, \sigma). \tag{16}$$

It has been shown empirically (Radd and Oertle 1973, Takashi et al. 1981) that there is a unique temperature  $T_1^{**}$  at  $n_1$  for a given  $\sigma > 0$  when an existing ice layer neither grows nor melts and f vanishes. This temperature  $T_1^{**}$  at the phase equilibrium of water is given as

$$\sigma = -\gamma T_1^{**}, \text{ if } f = 0.$$
 (17)

When an ice layer is steadily growing under a positive  $\sigma$ ,  $T_1$  is less than  $T_1^{**}$  (Nakano and Takeda 1994) and the frozen fringe  $R_1$  may be divided into two regions  $R_{10}(0 \ge T \ge T_1^{**})$  and  $R_{11}(T_1^{**} > T > T_1)$  as shown in Figure 4. At the phase equilibrium f vanishes and  $R_{11}$  also vanishes. From eq 4b we obtain

$$K_2/K_1 = \gamma$$
, in  $R_{10}$  if  $f = 0$ . (18)

It is easy to see that eq 12a is reduced to eq 17 at the phase equilibrium because of eq 18. An important question arises whether or not eq 18 holds true when f does not vanish. Since  $K_i$  (i=1,2) mainly depends on T and the composition of  $R_{10}$ , it is probable that eq 18 holds true for f>0 unless f significantly affects the composition. We will assume the validity of eq 18 regardless of f. With this assumption eq 4a becomes one step closer to the simpler models described earlier.

### **FUNCTION y**

The functional form of y depends on those of  $K_1$  and  $K_2$ . Based on available experimental findings (Horiguchi and Miller 1983, Nakano and Takeda 1994), we will assume that  $K_1$  and  $K_2$  are given as

$$K_1 = \begin{cases} K_0 & A \le T < 0 \\ K_0 (A/T)^{b_1} & A > T \end{cases}$$
 (19)

$$K_2 = \begin{cases} K_{20} & A \le T < 0 \\ K_{20} (A/T)^{b_2} & A > T \end{cases}$$
 (20)

Because of eq 4b  $K_0$  and  $K_{20}$  are related as

$$K_{20} / K_0 = \gamma$$
. (21)

The values of parameters in eq 19 and 20 determined empirically (Nakano and Takeda 1994) for Kanto loam are

$$K_0 = 1.77 \times 10^3 \text{ g/(cm d MPa)}$$
  
 $K_{20} = 1.98 \times 10^3 \text{ g/(cm d °C)}$   
 $b_1 = 0.520$   
 $b_2 = 1.04$   
 $A = -1.5 \times 10^{-4} \text{ °C}$ 

where d denotes day.

Using eq 19, 20 and 21 and eliminating  $T_1^{**}$ , we reduce eq 15 to

$$I_1 - I(\sigma) = \alpha_0 \, \delta_0 \, C_0 \, (1 + I_2) \gamma$$
 (22)

where

$$J_1 = C_1 \, y^{-\lambda_0} \tag{23a}$$

$$C_1 = \lambda_1^{-1} K_{20}^{\lambda_0} b_2 / (b_1 + 1)$$
 (23b)

$$J(0) = \lambda_1^{-1} (b_2 - b_1)$$
 (23c)

$$J(\sigma) = \begin{cases} \lambda_1^{-1} (b_2 - b_1) + z & A < T_1^{**} \\ \lambda_1^{-1} z^{\lambda_1} & A \ge T_1^{**} \end{cases}$$
 (23d)

$$C_0 = -(AK_{20})^{-1} > 0$$
 (23e)

$$J_2 = C_2(\alpha_0 \delta_0)^{-1} > 0$$
 (24a)

$$C_2 = -Ab_1(b_1+1)^{-1} > 0$$
 (24b)

$$z = C_3 \sigma \tag{24c}$$

$$C_3 = -(\gamma A)^{-1} > 0$$
 (24d)

$$\lambda_0 = \lambda_1/b_2 \tag{24e}$$

$$\lambda_1 = b_1 - b_2 + 1. \tag{24f}$$

For a special case where  $\lambda_1 = 0$ ,  $J_1$  and J are given as

$$J_1 = -b_2^{-1} \ln(y/K_{20})$$
 (25a)

$$J(0) = -b_1 b_2^{-1} (25b)$$

$$J(\sigma) = \begin{cases} -b_1 b_2^{-1} + z & A < T_1^{**} \\ -b_2^{-1} + \ln z & A \ge T_1^{**} \end{cases}$$
 (25c)

It is clear from eq 10 and 11 that  $\alpha_0$  and  $\delta_0$  must be positive because the mass flux  $f^*$  is positive at the formation of the final ice lens. When  $\alpha_0\delta_0$  is very small, eq 22 is reduced to

$$J_1 - J(\sigma) = C_0 C_2 y \tag{26}$$

It follows from eq 26 that y no longer depends on  $\alpha_0$  or  $\delta_0$  when  $\alpha_0\delta_0$  is very small. As shown in Appendix A, eq 22 generally possesses a unique positive root y. However, under certain conditions this unique root may not be physically meaningful. This implies that the formation of the final ice lens may not take place under such conditions.

When eq 22 possesses as a meaningful root, we will study the dependence of y on parameters,  $\alpha_0$ ,  $\delta_0$ ,  $\Delta P$  and  $\sigma$  below. We will write eq 22 as

$$J_1 - J(\sigma) = \alpha_0 \, \delta_0 \, C_0 \, y + C_0 \, C_2 y. \tag{27}$$

Differentiating eq 27 with respect to  $\alpha_0$  and  $\delta_0$ , we obtain

$$\frac{\partial y}{\partial \alpha_0} = \begin{cases} -\delta_0 C_0 y U^{-1} & \lambda_1 \neq 0 \\ -\delta_0 C_0 y U_0^{-1} & \lambda_1 = 0 \end{cases}$$
 (28a)

$$\frac{\partial y}{\partial \delta_0} = \begin{cases} -\alpha_0 C_0 y U^{-1} & \lambda_1 \neq 0 \\ -\alpha_0 C_0 y U_0^{-1} & \lambda_1 = 0 \end{cases}$$
 (28b)

where

$$U = \alpha_0 \delta_0 C_0 + C_0 C_2 + C_1 \lambda_0 y^{-(\lambda_0 + 1)}$$
 (29a)

$$U_0 = \alpha_0 \delta_0 C_0 + C_0 C_2 + b_2^{-1} v^{-1}$$
 (29b)

$$C_1\lambda_0 = K_{20}^{\lambda_0} / (b_1 + 1) > 0.$$
 (29c)

It follows from eq 28a and 28b that y is a decreasing function of both  $\alpha_0$  and  $\delta_0$ .

From eq 10 and 11, we obtain

$$\alpha_0 \delta_0 = K_0 \Delta P v^{-1}. \tag{30}$$

Using eq 30, we write eq 22 as

$$J_1 - J(\sigma) = K_0 C_0 \Delta P + C_0 C_2 y.$$
 (31)

Differentiating eq 31 with respect to  $\Delta P$ , we obtain

$$\frac{\partial y}{\partial \Delta P} = \begin{cases} -K_0 C_0 V^{-1} & \lambda_1 \neq 0 \\ -K_0 C_0 V_0^{-1} & \lambda_1 = 0 \end{cases}$$
 (32)

where

$$V = C_0 C_2 + C_1 \lambda_0 y^{-(\lambda_0 + 1)}$$
 (33a)

$$V_0 = C_0 C_2 + b_2^{-1} y^{-1}. (33b)$$

It follows from eq 32 that y is a decreasing function of  $\Delta P$ . This property of y is consistent with the empirical finding (the first inequality of eq 3).

Differentiating eq 27 with respect to  $\sigma$ , we obtain for  $\lambda_1 \neq 0$ :

$$\frac{\partial y}{\partial \sigma} = \begin{cases} -C_3 U^{-1} & A < T_1^{**} \\ -C_3^{\lambda_1} \sigma^{\lambda_1 - 1} U^{-1} & A \ge T_1^{**} \end{cases} . (34a)$$

When  $\lambda_1 = 0$ , we obtain

$$\frac{\partial y}{\partial \sigma} = \begin{cases} -C_3 U_0^{-1} & A < T_1^{**} \\ -\sigma^{-1} U_0^{-1} & A \ge T_1^{**} \end{cases} . (34b)$$

It is easy to find from eq 34a and 34b that y is a decreasing function of  $\sigma$ . This property of y is consistent with the empirical finding (the second inequality of eq 3).

Using experimentally determined parameters of Kanto loam (Nakano and Takeda 1994), we will show the behavior of y of Kanto loam below. For Kanto loam the values of  $\lambda_0$  and  $\lambda_1$  are 0.462 and 0.480, respectively. We will present the actual values of each term in eq 27:

$$J_1 = 4.75 \times 10 \ y^{-0.462}$$
 . (35a)

$$J(0) = 1.08$$
 (35b)

$$z = 5.95 \times 10^3 \,\mathrm{\sigma} \tag{35c}$$

$$J(\sigma) = \begin{cases} 1.08 + 5.95 \times 10^{3} \, \sigma & A < T_{1}^{\bullet} \\ 6.48 \times 10 \, \sigma^{0.480} & A \ge T_{1}^{\bullet} \end{cases}$$
 (35d)

$$\alpha_0 \delta_0 C_0 y = 3.37 \alpha_0 \delta_0 y \tag{35e}$$

$$C_0C_2y = 1.75 \times 10^{-4}y$$
 (35f)

where units of variables are  $y [g(cm \ ^{\circ}C \ d)^{-1}]$ ,  $\sigma [MPa]$ ,  $\alpha_0 [^{\circ}C \ cm^{-1}]$  and  $\delta_0 [cm]$ .

It is easy to find from eq 35e and 35f that the term  $C_0C_2y$  is much less than the term  $\alpha_0\delta_0C_0y$  unless  $\alpha_0\delta_0$  is very small. Neglecting this small term, we reduce eq 27 to

$$J_1 - J(\sigma) = \alpha_0 \delta_0 C_0 y. \tag{36a}$$

Using  $\Delta P$ , we write eq 36a as

$$J_1 - J(\sigma) = K_0 C_0 \Delta P \tag{36b}$$

When  $\sigma = 0$ , the term J(0) is much less than the other two terms of eq 36a or 36b. When J(0) is neglected, eq 36a and 36b are reduced to

$$y = (C_1/C_0)^{1/(\lambda_0+1)} (\alpha_0 \delta_0)^{-1/(\lambda_0+1)}$$
 (37a)

$$y = [C_1 / (K_0 C_0)]^{V \lambda_0} (\Delta P)^{-V \lambda_0}.$$
 (37b)

Suppose that we measure a set of values either  $(y, \alpha_0\delta_0)$  or  $(y, \Delta P)$  and that in y and  $\ln \alpha_0\delta_0$  (or  $\ln \Delta P$ ) are linear. If we know one of three parameters,  $b_1$ ,  $b_2$  and A, the remaining two parameters can be determined by eq 37a (or eq 37b). When  $\sigma \neq 0$ , the term  $J(\sigma)$  is not generally negligible and we must use eq 36a (or eq 36b) to determine an accurate dependence of y on  $\alpha_0\delta_0$  (or  $\Delta P$ ).

In Figure 5 we plotted experimental data, y vs.  $\alpha_0$  with  $\delta_0$  = 2 cm under  $\sigma$  = 0, 16.2, 48.7 and 195 kPa (Nakano and Takeda 1994) together with predicted y calculated by eq 36a. It is easy to see that ln y and ln  $\alpha_0$  are nearly linear when  $\sigma$  = 0. As  $\sigma$  increases the relationship between ln y and ln  $\alpha_0$  becomes nonlinear as anticipated.

### **CONCLUDING REMARKS**

Many models of ice segregation have been proposed in the past. However, the SP model proposed by Konrad and Morgenstern (1980, 1981) is one of few that were built on an empirical base. The SP model has been conveniently used to solve engineering problems. However, we need an accurate mathematical model that provides the functional dependence of SP on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) describing the properties of a given soil.

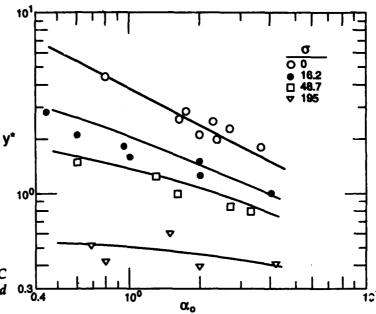


Figure 5. Experimental data of y [g(cm C d)<sup>-1</sup>] vs.  $\alpha_0$  [C/cm] together with predicted 0.3 y calculated by eq 36a.

In response to such a need Nakano (1990) proposed a new mathematical model called  $M_1$ . Efforts have been made to validate the  $M_1$  model by empirical findings and experimental data (Takeda and Nakano 1990, Nakano and Takeda 1991, Takeda and Nakano 1993, Nakano and Takeda 1994). In this report we presented the result of our efforts to validate the  $M_1$  model by using empirical findings that were used to build the SP model. It is important to mention that eq 1, 2 and 3 are empirical findings, not assumptions. If the  $M_1$  model is accurate, it must be able to explain these important empirical relationships. We have shown that the  $M_1$  model can explain these relationships.

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### **APPENDIX A: POSITIVE ROOT OF EQUATION 22**

We will show that eq 22 possesses a unique positive root y > 0. Using eq 27 that is equivalent to eq 22, we will introduce a function F(y) defined as:

$$F(y) = J_1 - \alpha_0 \delta_0 C_0 y - C_0 C_2 y - J(\sigma). \tag{A1}$$

It is clear that a root of eq 22 satisfies

$$F(y) = 0 (A2)$$

Differentiating F(y) with respect to y, we obtain

$$F'(y) = J_1'(y) - (\alpha_0 \delta_0 C_0 + C_0 C_2)$$
(A3)

where a prime denotes the differentiation with respect to y.

From eq 23a and 25a we obtain

$$J'(y) = \begin{cases} -\lambda_0 C_1 y^{-(\lambda_0 + 1)} & \lambda_1 \neq 0 \\ -b_2^{-1} y^{-1} & \lambda_1 = 0 \end{cases}$$
 (A4)

Since  $\lambda_0 C_1$  and  $b_2$  are positive, J'(y) is negative. Hence F'(y) is also negative, in other words, F(y) is a strictly decreasing function of y. We will examine the behavior of F(y) for two cases: Case 1,  $\lambda_1 \ge 0$ , and Case 2,  $\lambda_1 < 0$ .

For Case 1, F(y) approaches  $+\infty$  as y approaches 0 while F(y) approaches  $-\infty$  as y approaches  $+\infty$ . Since F(y) is a strictly decreasing function, eq 22 possesses a unique positive root. For Case 2, F(y) approaches  $-J(\sigma)$  as y approaches 0 while F(y) approaches  $-\infty$  as y approaches  $+\infty$ . When  $\lambda_1 < 0$ ,  $b_2 > b_1 + 1$ . Hence  $J(\sigma)$  is negative and eq 22 possesses a unique positive root. It may be noted parenthetically that  $A < T_1^{**}$  implies z < 1 because of eq 17

We have shown that eq 22 possesses a unique positive root regardless of  $\lambda_1$ . When an ice layer is growing, the temperature  $T_1$  must be less than  $T_1^{**}$ . This implies that y must be less than  $K_2(T_1^{**})$ , namely:

$$y < y^{**} = K_2(T_1^{**}) = K_{20} z^{-b_2}.$$
 (A5)

It should be noted that  $y^* = K_{20}$  when  $\sigma = 0$ . Equation A5 implies that  $F(y^{**})$  must be negative; that is

$$F(\mathbf{v}^{**}) = I_3(\mathbf{v}^{**}, \sigma) - C_4 z^{-b_2} < 0 \tag{A6}$$

where

$$J_3 = J_1(y^{**}) - J(\sigma)$$
 (A7)

$$C_4 = K_{20} C_0 (\alpha_0 \delta_0 + C_2) > 0.$$
 (A8)

The function  $J_3$  is given as:

$$J_3 = \begin{cases} b_1 b_2^{-1} > 0, & \lambda_1 = 0, \sigma = 0 \\ \ln z + b_1 b_2^{-1} - z & \lambda_1 = 0, A < T_1^{**} \\ b_2^{-1} > 0, & \lambda_1 = 0, A \ge T_1^{**} \end{cases}$$
(A9)

$$J_{3} = \begin{cases} b_{1} b_{2}^{-1} > 0, & \lambda_{1} = 0, \sigma = 0 \\ \ln z + b_{1} b_{2}^{-1} - z & \lambda_{1} = 0, A < T_{1}^{**} \\ b_{2}^{-1} > 0, & \lambda_{1} = 0, A \ge T_{1}^{**} \end{cases}$$
(A10)

It follows from eq A6, A9 and A10 that the condition of eq A6 is always satisfied when  $\lambda_1 \neq 0$  and  $A \geq T_1^{**}$ . However, for other cases it is not certain that eq A6 is satisfied.

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provides the functional deper tions in terms of well-defined such a need a mathematical n empirical findings and experi	ndence of SP on pertinent varial functions (or parameters) des model called M <sub>1</sub> was introduce mental data. In this report we	ture. An accurate mathematical model is needed the ables specifying given thermal and hydraulic condicribing the properties of a given soil. In response to d and efforts have been made to validate M <sub>1</sub> with will show that the functional dependence of SP on rical findings that were used to build the SP model.

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